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HOW GOOD ARE GLOBAL NEWTON METHODS?
Part 2

Allen Goldstein

September 1988

Approved for public release; distribution unlimited

Prepared for: Naval Surface Weapons Center
Dahlgren, VA 22448-5000

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NPS-53-88-010	5 MONITORING ORGANIZATION REPORT NUMBER(S) NPS-53-88-010
6a NAME OF PERFORMING ORGANIZATION Naval Postgraduate School; 6b OFFICE SYMBOL (If applicable) 53	7a NAME OF MONITORING ORGANIZATION Naval Surface Weapons Center
6c ADDRESS (City, State, and ZIP Code) Monterey, CA 93943	7b ADDRESS (City, State, and ZIP Code) Dahlgren, VA 22448-5000
8a NAME OF FUNDING/SPONSORING ORGANIZATION Naval Postgraduate School; 8b OFFICE SYMBOL (If applicable) 53	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN Direct Funding
8c ADDRESS (City, State, and ZIP Code) Monterey, CA 93943	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO; TASK NO.; WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
How Good Are Global Newton Methods. Part 2 - UNCLASSIFIED

12 PERSONAL AUTHOR(S) Allen A. Goldstein

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Technical Report	FROM 10/87 TO 8/88	1 September 1988	39

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Linear Programming; Computational Complexity, (jhl)

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Newton's method applied to certain problems with a discontinuous derivative operator is shown to be effective. A global Newton method in this setting is exhibited and its computational complexity is estimated. As an application a method is proposed to solve problems of linear inequalities (linear programming, phase 1). Using an example of the Klee-Minty type due to Blair, it was found that the simplex method (used in super-lindo) required over 2,000 iterations, while the method above required an average of 8 iterations (Newton steps) over 15 random starting values. Keywords: /

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a NAME OF RESPONSIBLE INDIVIDUAL Allen A. Goldstein	22b TELEPHONE (Include Area Code) (408) 646-2335	22c OFFICE SYMBOL 53Go

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

How good are global Newton methods. Part 2

A. A. Goldstein*

In the first part of this ms. we made some observations about global Newton methods on a simple class of mappings from a real separable Hilbert space H into H. The map F was assumed to be everywhere differentiable with a derivative D that was onto and satisfied $\mu\|h\| \leq \|D(x)h\| \leq \lambda\|h\|$, for all h $\in H$ and some $\mu > 0$. Relying on theorems of Nemerovsky and Yudin(1977), we showed that no formulation of a global Newton method could achieve better than linear convergence at a certain specified rate for every member of the class. Assuming that D was uniformly Lipschitz continuous with constant L it was easy to utilize the Kantorovich inequalities (1948). Points satisfying the Kantorovich inequalities are guaranteed to be a satisfactory initial point for Newton's method. By satisfactory, we mean that the sequence converges to a root at a quadratic rate, as will be seen below. Following a similar terminology of Smale(1986), such points will be called approximate roots.

We used the Kantorovich inequalities to formulate a coarse version of Smale's global Newton method to obtain the complexity of the algorithm in this setting. The Kantorovich condition is : if x_0 satisfies $\|D^{-1}(x_0)\|\,\|D^{-1}(x_0)F(x_0)\|\bar{L} = \beta\eta\bar{L} < .5$ then x_0 is an approximate root. Here $\bar{L}$ is the Lipschitz constant for D on the ball $B = \{x \in H : \|x - x_0\| \leq 2\|D^{-1}(x_0)F(x_0)\|\}$. We found that starting at x_0 we could achieve a point satisfying the Kantorovich inequalities in less than:

$$11.46\,[\|F(x_0)\|L/\mu^2\,]\,ln\,(1.443\,ln\,8Q)\ steps$$

where: L is the global Lipschitz constant, and Q = λ/μ the condition number. It is noteworthy that this algorithm is insensitive to the condition number. When the word 'steps' appears in this paper, as above, we mean the preceding formula to be rounded up to the nearest integer.

Some numerical experiments however were disappointing. The poor performance stemmed from our reliance on the Kantorovich inequality. This inequality is overconservative when

* supported by grant NPS LMC-M4E1

the dimension of H exceeds 1. In a test example even though we estimated $\bar{L}$ along only one ray (in the direction of the Newton step) and thus determined a $\bar{L}$ that was a lower bound, we found approximate roots where the Kantorovich constant $\beta\,\eta\,\bar{L}$ was several thousand instead of $< 1/2$. (See definition 1.5 below). For this reason we undertook to revise the Kantorovich inequalities, hoping to obtain better sufficient conditions. We shall do this by avoiding the Schwarz inequality at a key spot. In general we were guided by Kantorovich's original proof. A parameter K replaces $\bar{L}$ on the ball B. Estimates are given for the decrease in the norm of the vector valued function. This leads to the definition of a an ϵ approximate root. We believe this concept will be useful in the further development of global Newton algorithms. The estimates we give for the convergence rates can probably improved by a one dimensional analysis, as was done by Gragg and Tapia(1974) for the Kantorovich Theorem..

One consequence of the revised proof is that the derivative operator need not be continuous, although Lipschitz continuity is convenient in a neighborhood of each root. As an example we consider the problem of solving linear inequalities. (Phase 1 linear programming) By the use of Lagrange multipliers the method presented here can be extended to linear programming and other types of programs. We shall however only consider linear inequalities in this ms. The idea of using Newton's method for linear programming may be found in Smale(1986). Our technique will be to use a penalty function approach. In this setting we have a twice differential convex function to minimize, with jump discontinuities in the second derivative. For the code we refurbished an old global Newton method of our own. This was easy because it was already up and running. We tried the method on an example due to C. Blair(in Kortanek and Shi 1987) that is a version of the famous Klee-Minty example. This problem was suggested to us by Ken Kortanek, and we are thankful for his kind help. While Ken was visiting, he ran his scaling (modified Karmarker) algorithm on my computer for a benchmark in the 10 variable case. The robust 'Superlindo' by Linus Schrage was also used as a benchmark for these problems. For 8, 10, and 12 variables Superlindo took 35, 311, and 2236 iterations respectively. Our algorithm hardly changed in the number of steps versus dimension. Ken Kortanek remarked he noticed the same phenomena with their algorithm. In the appendix appears a list of 15 consecutive runs for the case n=12 with the components of the initial point being fed by a random number generator with values between -1000 and 1000. The number of steps until termination ran between 6 Newton steps plus 2 gradient steps, and 12 Newton steps plus 3 gradient steps.

The average was 8 Newton steps and 19/15 gradient steps. Full double precision accuracy was achieved. The Kortanek-Shi algorithm in the 10 variable case was almost as fast as ours for this case, but it achieved only 3 significant figure accuracy.

These spectacular results were unexpected. Unlike the Smale global Newton method, which we as yet have not implemented, the method we used is sensitive to the condition number of the system. See 3.0 below.

There remains to test the algorithm on a general mix of problems to see if it is worth adding to the armamentarium of linear programming. Thus the question of whether we have a viable method is not settled at this time.

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In what follows, the Kantorovich inequality will be sharpened. Let f be Frechet differentiable mapping between real Banach spaces E and F. Let $f'(x)$ denote the derivative at x and $f'_{-1}(x)$ its inverse. The Kantorovich inequality states that if x_0 is given such that f' is Lipschitz continuous with constant K on the ball $\{x \in E : \|x - x_0\| \leq 2\|f'_{-1}(x_0)f(x_0)\| = 2\,\eta_0\}$, if $\|f'_{-1}(x_0)\| = \beta_0$ and if $\eta_0\beta_0 K < 1/2$, then x_0 is an approximate root. This means that the sequence generated by Newton's method will converge quadratically to a root at least as fast as the rate estimated by the theorem below.

THEOREM 1.0 Let f be a map between real Banach spaces E and F. Assume f is Frechet differentiable on an open subset E' of E. Let $x_0 \in E'$ be given such that $(f'(x_0))^{-1} = f'_{-1}(x_0)$ exists. Set

$$\eta_0 = \|f'_{-1}(x_0)f(x_0)\|$$

$$S = \{\, x \in E : \|x - x_0\| \leq 1.68\,\eta_0 \,\}. \; \textit{Assume that } S \subset E'$$

Set

$$\beta_0 = \|f'_{-1}(x_0)\| \;, \quad \textit{and let}$$

$$S' = \{\, x \in S : \|f'_{-1}(x)\| \leq 2.3\,\beta_0 \}.$$

Finally set:

$$K = \left\{ sup \frac{\|f'_{-1}(x)(f'(x) - f'(\xi))\|}{\|f'_{-1}(x)\| \; \|f'_{-1}(x)f(x)\|} : x \in S' \;,\; \xi = x + tf'_{-1}(x)f(x) \textit{ and } t \in (0,1) \right\}.$$

If $\eta_0\beta_0 K = h_0 \leq 1/3$, then x_0 is an approximate root.

PROOF By hypothesis x_1 is well defined. Let $H_1(x_0, x_1) = H_1 = f'_{-1}(x_0)f'(x_1) = I - f'_{-1}(x_0)(f'(x_0) - f'(x_1))$. H_1 maps E into itself. Our hypotheses imply that $\|f'_{-1}(x_0)(f'(x_0) - f'(x_1))\| \leq h_0 \leq 1/3$, whence H_1 has an inverse. We have the estimate $\|(H_1)^{-1}\| = (1 - h_0)^{-1} \leq 3/2$. Also $\|H_1\| \leq 4/3$. Observe that $f'(x_0)H_1 = f'(x_1)$ and $(H_1)^{-1}f'_{-1}(x_0) = f'_{-1}(x_1)$. Thus $f'_{-1}(x_1)$ exists and x_2 is well defined.

Let $\beta_1 = \|f'_{-1}(x_1)\|$ and $\eta_1 = \|x_1 - x_2\|$. Thus $\beta_1 = \|f'_{-1}(x_1)\| \leq 1.5\;\|f'_{-1}(x_0)\|$. Let F_1 be defined by the formula $F_1(x) = x - f'_{-1}(x_0)f(x)$. Since $F_1(x_0) = x_1$ and $F_1(x_1) = x_1 - f'_{-1}(x_0)f(x_1)$,

$$f'_{-1}(x_0)f(x_1) = F_1(x_0) - F_1(x_1).$$

By the generalized mean value theorem of Graves(1927),

$$\|F_1(x_1) - F_1(x_0)\| \leq sup\{ \ \|F_1'(\xi)\| : t \in (0,1) \ and \ \xi = tx_0 + (1-t)x_1 \ \}\|x_1 - x_0\|.$$

But $\|F_1'(\xi)\| = \|I - f_{-1}'(x_0)f'(\xi)\| = \|f_{-1}'(x_0)(f'(x_0) - f'(\xi))\| \leq K\eta_0\beta_0$. Thus:

$$\|f_{-1}'(x_1)f(x_1)\| = h_0\,\eta_0,$$

$$\eta_1 \ = \ \|f_{-1}'(x_1)\,f(x_1)\| \ = \|H_1^{-1}\,f_{-1}'(x_0)\,f(x_1)\| \leq \ h_0\eta_0(1-h_0)^{-1}$$

and

$$h_1 = \beta_1\eta_1 K \leq (1-h_0)^{-2}h_0^2 \leq 9h_0^2/4 \leq 1/4$$

At this juncture our estimate will differ from that of Kantorovich by a factor of 1/2 that will appear in the recursion for h_n. Kantorovich gains a factor of 1/2 because of the Lipschitz condition (or 2nd differentiability condition) on H. Clearly x_1 *and* x_2 belong to S, and also to S'. Now define in turn H_2, β_2, η_2, F_2, *and* h_2, mutatis mutandis, to obtain β_2, η_2, *and* h_2 in terms of β_1, η_1, *and* h_1.

For k=1,2,3,...,n, assume that $\beta_k\eta_k K \leq 1/3$ and $x_k \in S'$. Assume moreover that the recursion formulae:

$$\beta_n = \beta_{n-1}(1-h_{n-1})^{-1} \qquad \eta_n = h_{n-1}\eta_{n-1}(1-h_{n-1})^{-1} \qquad h_n = h_{n-1}^2(1-h_{n-1})^{-1}$$

hold for k=1,2,3,...,n. Since these recursion formulae hold also for k = n+1, we have the following sequence for $\{h_k\}$

$$\frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{64}, \ \frac{1}{3968}, \ \ldots, \ \frac{1}{(\frac{1}{h_n}-1)^2}$$

Since:

$$\beta_{n+1} \leq \beta_0\left(\frac{1}{(1-h_0)}\,\frac{1}{(1-h_1)}\,,\ldots,\,\frac{1}{(1-h_n)}\right)$$

$$\eta_{n+1} \leq \eta_0\left(\frac{h_0}{1-h_0}\,\frac{h_1}{1-h_1}\,\frac{h_2}{1-h2}\,,\ldots,\,\frac{h_n}{1-h_n}\right)$$

and

$$\|f'(x_{n+1})\| \leq \|f'(x_0)\|\,(1+h_0)\,(1+h_1)\,,\ldots,\,(1+h_n)$$

We get from above the formulae:

$$\beta_{n+1} \leq \beta_0 \left(\frac{3 \cdot 4 \cdot 9 \cdot 64 \cdot 3969 \cdot 1575024 \cdot, ..., \cdot 1}{2 \cdot 3 \cdot 8 \cdot 63 \cdot 3968 \cdot 1575023 \cdot, ..., \cdot (1 - h_n)} \right) \leq 2.3\, \beta_0, \; for \; n = 1, 2, 3, ...$$

$$\eta_{n+1} \leq \eta_0 \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{8} \cdot \frac{1}{63} \cdot \frac{1}{3967}, ..., \frac{h_n}{(1 - h_n)} \right)$$

$$\|f'(x_n)\| \leq \|f'(x_0)\| \left(\frac{4 \cdot 5 \cdot 10 \cdot 65 \cdot 3970 \cdot, ..., \cdot (1 + (h_n)^{-1})}{3 \cdot 4 \cdot 9 \cdot 64 \cdot 3969, ..., \cdot (h_n)^{-1}} \right)$$

$$\|f'(x_{n+1}\| \leq 1.9 \|f'(x_0)\| \; for \; n = 1, 2, 3......$$

To complete the induction we show that $x_{n+1} \in S'$. The triangle inequality gives:

$$\|x_{n+1} - x_0\| \leq \eta_0 + \eta_1 +, ..., \eta_n$$

$$\leq \eta_0 \left(1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{48} + \frac{1}{3024} +, ..., \prod_{k=1}^{n} \frac{h_k}{1 - h_k} \right) \leq 1.68\, \eta_0 \; for \; n = 1, 2, 3...$$

Thus $x_n \in S$. Since $\beta_{n+1} \leq 2.3\beta_0$ by the above inequality, the induction is completed.

Similarly we have:

$$\|x_{n+p+1} - x_n\| \leq \eta_n + \eta_{n+1} + \eta_{n+p} \; < \; 1.68\eta_n.$$

Since $\{\eta_n\}$ converges to 0, the sequence $\{x_n\}$ is Cauchy with limit, say x^*. Since $f(x_n) = f'(x_n)(x_{n+1} - x_n)$, $\{f(x_n)\}$ converges to 0, and $f(x^*) = 0$.

We now estimate the rate of convergence. We have, at worst, that $h_n \leq \frac{4}{9}\left(\frac{9}{4}h_0\right)^{2^n} \leq \frac{4}{9}\left(\frac{3}{4}\right)^{2^n}$. The recursion formula for η_n implies that

$$\eta_n \leq h_{n-1} h_{n-2} ... h_0 \eta_0 (1 - h_{n-1})^{-1} (1 - h_{n-2})^{-1} ... (1 - h_0)^{-1}.$$

Also $\eta_n \leq \left(\frac{4}{9}\right)^n \left(\frac{9}{4}h_0\right)^{2^{n-1}} \left(\frac{9}{4}h_0\right)^{2n-2} ... \left(\frac{9}{4}h_0\right)\eta_0$. Since $\sum_{k=1}^{n} 2^{n-k} = 2^n - 1$,

$$\frac{\eta_n}{\eta_0} \leq \left(\frac{4}{9}\right)^n \left(\frac{9}{4}h_0\right)^{2^n - 1}.$$

Since $\|f(x_n)\| = \|f'(x_n)\|\, \eta_n$, while $\|f'(x_n)\| \leq 1.9\, \|f'(x_0)\|$, we have the estimates

$$(I) \quad \frac{\|f(x_n)\|}{\|f(x_0)\|} \leq 1.9\, \frac{\eta_n}{\eta_0} \qquad and \qquad (II) \quad \frac{\|x_n - x^*\|}{\|x_1 - x_0\|} \leq 1.68\, \frac{\eta_n}{\eta_0}.$$

REMARK 1.1 Assume the hypotheses of the theorem with suitable changes in S and S'. Take $h_0 < 1/2$. Then x_0 is an approximate root.

REMARK 1.2 The theorem does not require the continuity of f'. Let f(x) = 4x, if $\infty < x \leq 1$ and f(x) = 6x -2, if x > 1. Then every point is an approximate root for f. For example let $x_0 = 100$. Then $h_0 = 1/3$, $x_1 = 1/3$, $h_1 = 0$, and $x_2 = 0$.

REMARK 1.3 Let S, S' and K be defined as above. Assume that f' is Lipschitz continuous with constant L on S. Let $h_k = \sup\{\|f'_{-1}(x_k)(f'(x_0) - f'(\xi)\| : x \in S' \; and \; \xi = x_k + tf'_{-1}(x_k)f(x_k), t \in (0,1)\}$. Assume the Kantorovich inequality holds. Then $h_k \leq \beta_k \eta_k L \leq 1/3$. Whence $h_k/\beta_k\eta_k = K \leq L$. We have replaced L by K. Another hypothesis to ensure the boundedness of K (which we've assumed outright) is to require that f' be bounded on S' and, for any root x* of f there is a neighborhood N(x*) e such that f' is Lipschitz continuous on N(x*)$\cap S'$.

REMARK 1.4 The Kantorovich type theorems are important for the insight they furnish, but in general are not helpful in concrete instances. For example, the above theorem and the original Kantorovich theorem are dependent upon the knowledge of the constant K or L, but the determination of upper bounds for these quantities is not trivial. Moreover the conditions given by these inequalities is sufficient but not in general necessary for a point to be an approximate root. Thus the determination of approximate roots is usually not practical with these theorems. Using the rough estimates furnished by local application of the modified Kantorovich inequality we propose proceeding by trial.

DEFINITION 1.5 Given $\epsilon > 0$, let n be the smallest integer such that

$$1.9\left(\frac{4}{9}\right)^n \left(\frac{3}{4}\right)^{2^n-1} < \epsilon.$$

A point x_0 is an ϵ-approximate root for the mapping f if the Newton sequence $\{x_i\}$ starting at x_0 is well defined for i=1,2,..., n, and

$$\frac{\|f(x_n)\|}{\|f(x_0)\|} < \epsilon,$$

For example if $\epsilon = 10^{-18}$ then n = 7. This test follows from the estimates I and II of the above theorem.

LEMMA 2.0 Let A be a positive definite matrix. Let μ be its least eigenvalue and λ its greatest. Let Q = λ/μ. Then:

$$\frac{[Ax,x]}{\|Ax\|\,\|x\|} \geq \frac{2}{Q^{\frac{1}{2}}}\frac{Q}{(Q+1)}.$$

The bound is the best possible.

PROOF. Assume A has been diagonalized, and let

$$[Ax,x]/\mu \;=\; x_1^2 + c_2x_2^2 +, ..., c_nx_n^2.$$

We seek to minimize $f(x) = (x_1^2 + c_2x^2 +, ..., c_4x^4)/(x_1^2 + c_2^2x_2^2 =, ..., c_4^2x_4^2)^{\frac{1}{2}}$, subject to $\|x\| = 1$. The reason for restricting to polynomials of degree 4 in x_i^2 will become apparent in what follows.

Set $x_1^2 = 1 - x_2^2 - x_3^2 - x_4^2$. Using this equation to eliminate x_1^2 we get:

$$f(x) = \frac{1 + (c_2-1)x_2^2 + (c_3-1)x_3^2 + (c_4-1)x_4^2}{(1 + (c_2^2-1)x_2^2 + (c_3^2-1)x_3^2 + (c_4^2-1)x_4^2)^{\frac{1}{2}}}$$

Let P denote the numerator of the above fraction and $Q^{\frac{1}{2}}$ the denominator. Let x = $(x_2^2,\ x_3^2,\ x_4^2)$ and

$$A \;=\; (c_2 - 1,\ c_3 - 1,\ c_4 - 1)$$

and

$$B \;=\; (c_2^2 - 1,\ c_2^2 - 1,\ c_4^2 - 1)$$

The components of the equation $\nabla f(x) \;=\; 0$ may be written as :

$$x_2(c_2-1)[2B-(c_2+1)A,\ x] \;=\; x_2(c_c-1)(c_2-1) \qquad (a)$$

$$x_3(c_3-1)[2B-(c_3+1)A,\ x] \;=\; x_3(c_3-1)(c_3-1) \qquad (b)$$

$$x_4(c_4-1)[2B-(c_4+1)A,\ x] \;=\; x_4(c_4-1)^2 \qquad (c)$$

If we assume distinct eigenvalues and that x_2, x_3 *and* x_4 are non-zero we may cancel appropriately and obtain from (a), (b), and (c), a system of 3 linear equations of rank 2,

since the rows of the system are linear combinations of A and B. Thus one unknown say x_2 or x_3 (but not x_4) may be set to 0. Similarly if we had n variables only one variable among x_2 *and* x_{n-1} would be non-zero. If we set $x_2 = 0$, then the following system obtains:

$$(c_3^2 - 1)x_3^2 \ + \ (c_4 - 1)(c_4 - c_3)x_4^2 \ = \ c_3 - 1$$

$$(c_3^2 - 1)x_3^2 \ + \ (c_4^2 - 1)x_4^2 \ = \ c_4 - 1$$

One finds that

$$x_4^2 = \frac{(c_3^2 - 1)(c_4 - c_3)}{-(c_3 - 1)(c_3 + 1)(c_4 - 1)(c_4 - c_3)} < 0.$$

Since x_4^2 is negative, it is inadmissible. Thus x_3^2 must also vanish. Suppose now that $x_3 = 0$ but $x_2 \neq 0$. We again get $x_4^2 < 0$. Thus for these cases we must set x_2 *and* x_3 to 0.

If $c_2 > 1$ and $c_4 > c_2$ then the rank of $\{A, B\} = 2$ This case is treated just as the case above of distinct roots. If all the eigenvalues are equal, the ratio in the statement of the lemma is 1, verifying the claim for that case. For all other cases the rank of $\{A, B\} = 1$, and we may set x_2 *and* x_3 to 0.

There remains to carry out the minimization. The equation

$$(c_4^2 - 1)x_4^2 = c_4 - 1 \qquad \textit{implies} \qquad x_4^2 = \frac{1}{Q+1}$$

$$\textit{and}$$

$$f(x) \ = \ \frac{2\frac{Q}{Q+1}}{Q^{\frac{1}{2}}}$$

Algorithm 3.0

Let f be a twice differentiable function defined on real euclidean space E_n. Assume that f is convex and that it has bounded level sets. Given an arbitrary point x_0 let S denote the level set $\{x \in R_n : f(x) \leq f(x_0)\}$. Assume that the hessian of f at x, which we call H(x), has an inverse at x_0, denoted by $H^{-1}(x_0)$. At the kth step of the algorithm assume that $H^{-1}(x_k)$ exists. Let

$$g(x, \gamma) = \frac{f(x) - f(x - \gamma H^{-1}(x)\nabla f(x))}{\gamma[H^{-1}(x)\nabla f(x),\ \nabla f(x)]}$$

Choose $\delta \in (0,1/2]$ and choose $\gamma_k > 0$ so that $\delta \leq g(x_k, \gamma_k) \leq 1 - \delta$, taking $\gamma_k = 1$, if possible. Set $x_{k+1} = x_k - \gamma_k H^{-1}(x_k) \nabla f(x_k)$. Let $S' = \{x_k \in S : k = 1, 2, 3, ...\}$. Assume that the spectrum of H(x) is bounded above on S' by ΛQ, and below by Λ. Let

$$h = sup\Big\{\|H^{-1}(x)(H(x) - H(\xi))\| : x \in S' \text{ and } \xi = x + tH^{-1}(x)\nabla f(x),\ t \in [0,1]\Big\}.$$

(a) Assume that K =

$$sup\left\{\frac{\|H^{-1}(x)(H(x) - H(\xi))\|}{\|H^{-1}(x)\|\,\|H^{-1}(x)\nabla f(x)\|} : x \in S' \text{ and } \xi = x + tH^{-1}(x)\nabla f(x),\ t \in [0,1]\right\} < \infty.$$

Let N be the least integer exceeding:

$$\frac{f(x_0) - minf}{\Delta} \qquad where \qquad \Delta = \frac{\delta^2 \Lambda^3}{(1 + 2hQ^{1/2})K^2 Q^{1/2}}$$

Then an approximate root of $\nabla f(x)$ will be achieved in at most N steps.

(b) Let

$$\Delta_1 = \frac{\delta^2 (1-\delta)^2 \Lambda^3}{(1 + 2h\, Q^{\frac{1}{2}}\, K^2\, Q^{\frac{3}{2}}}$$

. Let M be the least integer exceeding:

$$\frac{f(x_0) - minf}{\min\{\Delta, \Delta_1\}}$$

. Then in at most M steps an approximate root will be achieved, and all subseqent steps will be Newton steps.

PROOF (a) Assume in what follows that x $\in S$. The change in f do to the step $\gamma s(x) = \gamma H^{-1}(x) \nabla f(x)$ may be estimated by Taylor's formula as:

$$(*)\ \Delta(x) = f(x) - f(x - \gamma\, s(x)) = \gamma \left[1 - \frac{\gamma}{2} - \frac{\gamma}{2} \frac{[s(x), (H(\xi) - H(x))\, s(x)]}{[\nabla f(x), s(x)]}\right] [\nabla f(x), s(x)]$$

or

$$(**) \qquad \Delta(x) \;=\; \gamma g(x, \gamma)[\nabla f(x), s(x)]$$

Here ξ lies between x and x - $\gamma\, s(x)$. By the above formula we see that the right hand limit of $g(x, \gamma)$ *at* $\gamma = 0$ is 1. Since S is bounded, for some $\hat{\gamma}$, $g(x, \hat{\gamma}) = 0$. This shows that

γ may be chosen as claimed. We see also that $\{f(x_k)\}$ is a decreasing sequence so that $x_k \in S$ for k=1,2,3,......

We aim now to find a lower bound for $\Delta(x)$. In the equation (*) above, the third term in the large brackets is

$$\geq -\frac{\gamma}{2}\frac{\|\nabla f(x)\|\,\|H^{-1}(x)(H(x)-H(\xi)\|\,\|s(x)\|}{[\nabla f(x), H^{-1}(x)\nabla f(x)]}.$$

Using lemma 2.0 we get the inequalities

$$1-\frac{\gamma}{2}+\frac{\gamma}{2}\alpha\, h(x)\, Q^{\frac{1}{2}} \;\geq\; g(x,\gamma) \;\geq\; 1-\frac{\gamma}{2}-\frac{\gamma}{2}\alpha\, h(x)\, Q^{\frac{1}{2}}.$$

Where h(x) = sup $\{\|H^{-1}(x)(H(x)-H(\xi)\| : \xi = x + ts(x)\ with\ t \in [0,1]\}$. and:

$$\alpha = \frac{Q+1}{2Q}.$$

so that $.5 < \alpha \leq 1$. Since $g(x,\gamma) \leq 1-\delta$ we thus obtain the lower bound

$$\gamma \;\geq\; \frac{2\delta}{1+\alpha\, h(x) Q^{\frac{1}{2}}}$$

If $\{\|\nabla f(x_k)\|\}$ does not converge to 0, an infinite subsequence of it is bounded away fr m 0. Since $[\nabla f(x), s(x)] \;\geq\; \|\nabla f(x)\|^2/Q\Lambda$, and $\gamma_k g(x_k,\gamma_k)$ is also bounded away from 0 $f(x_k)$ tends to $-\infty$. This contradiction establishes that the sequence $\{\nabla f(x_k)\}$ converge to 0. Moreover every cluster point of the set S' minimizes f, and $\{f(x_k)\}$ converges to min f.

If x does not satisfy our condition for an approximate root then $\|H^{-1}(x)\|\,\|s(x)\|\,K \;\geq$ 1/2. By the lemma $[\nabla f(x), s(x)] \;\geq\; (Q^{1/2}\alpha)^{-1}\|s(x)\|\,\|\nabla f(x)\|$. Consequently, $\|s(x)\| \;\geq$ $1/2K\,\|H^{-1}(x)\|$, and $\|\nabla f(x)\|\,\|s(x)\| \;\geq\; 1/4K^2\Lambda^3$. Thus a lower estimate of (**) is

$$\Delta(x) \;\geq\; \frac{\delta^2}{(1+2h\,Q^{\frac{1}{2}}\,4\Lambda^3\,K^2\,Q^{\frac{1}{2}}} \;=\; \Delta$$

We may similarly find a number of steps that will guarantee that an approximate root has been reached and that the above algorithm will always produce subsequent Newton steps. We do this by observing that $.5K\|s(x)\|\Lambda\alpha Q^{1/2} \leq\; .5-\delta$ implies that $.5Kh(x)\alpha Q^{1/2} \;\leq$

.5 − δ. And this implies that $\delta \leq \; g(x,1) \; \leq \; 1-\delta$. Thus we shall count the steps for which $\|s(x_k)\|$ exceeds $(1-2\delta)/\alpha K \Lambda Q^{1/2}$. As above we find that

$$\Delta(x) \; \geq \; \frac{\delta^2(.5-\delta)^2\Lambda^3}{(1+2h\,Q^{\frac{1}{2}}\,K^2\,Q^{\frac{3}{2}}} \; = \; \Delta_1$$

We now "count" the steps. Since $\Delta(x_k)$ is bounded away from zero, say by Δ, we have $f(x_0) - f(x_k) \geq \; k\left(\frac{1}{k}\,\sum_{i=0}^{k}\Delta(x_i)\right) \; \geq \; k\Delta$. Set

$$N = \frac{f(x_0) - min f}{\Delta}$$

Set

$$M = \frac{f(x_0) - min f}{min\{\Delta, \Delta_1\}}.$$

If $k \; = \; N$, *and* x_k is not an approximate root, we have a contradiction.

REMARK 3.1 The results above are disappointing in that the cost of the algorithm is sensitive to the condition number Q, in spite of the fact that Newton directions are taken. (Recall that Smale's global Newton method was not sensitive to the condition number). Indeed the gradient method is less sensitive to Q than this method. We may classify this algorithm as "greedy" because it is trying to decrease f at each iteration. On the plus side, the algorithm is easily implemented.

The most robust result over the class of strongly convex functions is due to Nesterov. A simple, easily coded algorithm using combinations of gradient steps will drive $f(x_n)/f(x_0)$ to less than ϵ in less than

$$\frac{4\sqrt{Q}}{ln 2}\, ln\left(\frac{1}{\epsilon}\right) \; steps$$

. By the theory of Nemerovsky and Yuden, for some positive number c no algorithm can do better than the following number of steps for every strongly convex function of condition number Q:

$$c\frac{min(n,\sqrt{Q})}{ln\, min(n,\sqrt{Q})}\, ln\frac{1}{\epsilon}.$$

For the case when n > $\sqrt{Q}$ we can assert that Nesterov's method is to within a slowly changing multiplicative factor essentially optimal over the class of strongly convex functions. This method and its generalizations are being studied by Osman Guler at U. of Chicago, School of Business.

Consider the Smale global Newton method applied to minimizing strongly convex functions of condition number Q. This method which is sensitive to a Lipschitz constant for the Hessian would not be a candidate for optimality on the set of strongly convex functions. The reason is that there exist strongly convex functions with condition number Q that have Hessians with arbitrarily large Lipschitz constants. However for the multitude of natural problems with bounded Lipschitz 2nd derivatives, or bounded values of the constant K, we should expect better results for large condition numbers from the Smale global Newton method, than with an algorithm of the Nesterov type.

We now digress to the problem of linear inequalities. Let A be an m by n matrix of rank n, and b an m by 1 matrix. Denote the ith row of A by A^i. Set

$$R(x) = Ax - b.$$

We seek a solution of the system of inequalities :

$$R_i(x) \ \leq \ 0 \qquad 1 = 1, 2, ..., m.$$

Or, if this system has no solution, to establish its inconsistency. Let G(x) $= max\{R_i(x) : 1 \leq i \leq m\}$. We assume that G is bounded below, and consequently, that G has bounded level sets. To ensure that this is the case, a phony half-space may be addeα to our system of inequalities. Let $A_0 = -\sum_{j=1}^{m} A_i$. Let $R_0(x) = [A_0, x] - b$. Let $G*(x) = max\{R_i(x) : 0 \leq i \leq m\}]$. Let z(b) = argmin G*. If the original system is consistent then for b sufficiently large G(z(b))$\leq$ 0.

We shall employ the penalty function:

$$\sum_{i=1}^{m} [max(0, R_i(x))]^p \qquad with\ p \geq \ 2$$

We note that for $p \geq 2$, F is convex. If p=2, F' is continuous, and F'' has jump discontinuities. If p > 3 then F''' is continuous. Our numerical experiments using F to solve inequalities gave most satisfactory results with p=2, even though more smoothness is obtained with larger values of p. Thus in what follows we shall take p=2. We assume that F has bounded level sets.

Let $I^+(x)$ denote the set of all indices from $\{1, 2, 3, ..., m\}$ for which $R_i(x) > 0$ when $i \in I^+(x)$. If $dim\{A_i : i \in I^+(x)\} < n$, then $F''(x)$ does not have an inverse. We seek the following construction.

CONSTRUCTION B 4.0

Assume that at x_k the dimension of the set of gradients of the actice residuals is less than n. That is, dim $\{A_i : 1 \leq i \leq q\} \leq n$ Take h $\neq$ 0 such that $[A_i, h] = 0$, $1 \leq i \leq q$. Let $\epsilon_k = .5(F(x_{k-1} - F(x_k)$. Find the smallest t such that $\sum\{R_i^2(x + th) : q + 1 \leq i \leq m \ and \ R_i > 0\} = \epsilon_k^2 / 2(m - q)$ Set $x_{k1} = x_k + th$. Repeat this process if necessary. After at most m-q steps we obtain x_{k+1}.

ALGORITHM 5.0 Take x_0 arbitrarily in E_n. If $dim\{A_i : i \in I^+(x_0)\} < n$ use (4.0) to find x_1 with $dim\{A_i \in I^+(x_1)\} = n$. At the kth iteration we are given x_k and a non-singular $H(x_k)$. Choose γ_k such that $.75 \geq g(x_k, \gamma_k) \geq .25$, taking $\gamma_k = 1$, if possible. Set $\bar{x}_{k+1} = x_k - \gamma_k H^{-1}(x_k)\nabla f(x)$. If H has an inverse at $\bar{x}_{k+1}$ then set $x_{k+1} = \bar{x}_{k+1}$ Otherwise update $\bar{x}_{k=1}$ with the construction (4.0) or (4.1) to obtain x_{k+1}.

CLAIM 5.1 Assume the hypotheses of (4.0). The algorithm is well defined and will terminate in a finite number of steps, with a positive value of F if the system is inconsistent and with F $=$ 0 if the system is consistent. For the latter case the argument of F is a solution of the inequalities.

PROOF Let X $= \{x \in E_n : [A^i, x] \leq b_i,\ 1 \leq i \leq m\}$. Assume that X is not empty. Let w denote the number of vertices of the solution set X, and let v_s , s=1,2,...,w denote the vertices. Each v_s lies in the intersection of at least n hyperplanes supporting X. Any point in X is on the right side ($R_i(x) \leq 0$) of these hyperplanes. Stated otherwise: $[A^i, v_s] = b_i,\ i \in I_s,\ card\, I_s \geq n$. Let $C(v_s) = \{x \in E_n : [A^i, x] > b_i, i \in I_s\} = \{x \in E_n : [A^i, x - v_s] > 0, i \in I_s\}$,and let $\bar{C}(v_s)$ denote its closure. Let B_s be a ball of radius r_s centered at v_s. If some hyperplane $[A^i, x] - b_i$ such that $i \in I \sim I_s$ meets $C(v_s)$ then there exists a minimal value of r_s such that $B_s \cap \bar{C}(v_s)$ meets the closest hyperplane $[A^i, x] - b_i$ such that $i \in I \sim I_s$. Points in $B_s \cap C(s) \sim \{v_s\}$ are on the wrong side of the same hyperplanes, n or more in number. In all the above sets s is understood to range over 1,2,...,w.

It follows from the proof of (3.0) that the above algorithm generates a sequence $\{x_k\} \in S$ (the level set of F at x_0) such that $F(x_k)$ converges downward to min F. We claim that the only cluster points of $\{x_k\}$ are the vertices v_s, $s = 1, 2, ..., w$. The sets $I^+(x_k)$ are collections of n or more out of m indices that are repeated infinitely often. Thus at least one of the collections must be frequently repeated. Let $\{x_{k_i}\}$ denote a subsequence converging to a limit v. Take a thinner subsequence if necessary such that only one index set is represented. Since F(v) $=$ 0, it folows that $v \in X$. Since $\{x_{k_i}\}$ is on the wrong side of at least n hyperplanes at points arbitrarily close to X, the index set I must be one of the sets I_s, s=1,2,...,w. Let $\bar{k}$ denote the least value of k such that $x_{\bar{k}} \in B_s \cap C(s)$ for some s =1,2,...,w. At most one more step at $x_{\bar{k}}$ terminates the process, because a Newton step minimizes the restriction of F to $B_s \cap C(v_s)$ –a quadratic function,– in one step.

Assume now that X is empty. A necessary condition that a system of inequalities be inconsistent is that 0 belongs to the convex hull of the rows of the matrix A. Let x* minimize F. We claim that x* is unique, $card\, I^{+}(x*) \geq n+1$, and 0 belongs to the convex hull of the rows of A. If $card\, I^{+}(x*) \leq n$ choose h so that $[A^{i}, h] = -1, i \in I^{+}(x*)$. Then for some h sufficiently small F(x*) can be decreased. Since x* is a strict local minimum it is unique because of the convexity of F.

Let B be a ball centered at x* with the property that $I^{+}(x) = I^{+}(x*)$ for all x $\in B$. This existence of such a ball follows from the continuity of $R_i(x)$ for each i, $1 \leq i \leq m$, and because $R_i(x*) \neq 0$ for all i $\in I^{+}(x*)$. Because x* is unique the sequence $\{x_k\}$ converges to it. Thus for some least k = $\bar{k}$ $x_{\bar{k}} \in B$. Thus the solution occurs here or at the next step.

Notice that the parameter K of (3.0) is actually finite. Consider the denominator of K for k=0,1,2,...,$\bar{k}$. Since $\|s(x)\| \geq \|\nabla f(x)\|/Q\mathcal{L}$ the denominator is always positive. (If $\nabla f(x) = 0$ we have a solution) One more step at $x_{\bar{k}}$ terminates the process. The Newton step minimizes the quadratic function (restriction of F to $B_s \cap C(v_s)$ in one step. Note that the numerator of K for this last step is 0, while the denominator is positive.

NUMERICAL COMPUTATIONS 6.0

The Blair example may be found in Kortanek and Shi. The coefficients for the case n=8 are written out (Example 2a, p55) and a program is given to generate the coefficients for any n. For n = 12, the case for which we offer extensive calculations, the Blair problem is a linear programming problem (LP) of the form: minimize L(x) subject to

$$R_i(x) \leq 0 \qquad 1 \leq i \leq 24$$

of which 12 inequalities constrain x to lie in the first orthant. The coefficients of the system are integers between 1 and 305,175,780. Thus the system is very poorly conditioned. The problem has a unique solution $\hat{x}_i = 0$, i=1,2,...,11, $\hat{x}_{12} = 1$, with $L(\hat{x}) = 305175780$. The system we solve contains following inequalities :

$$r_i(x) \leq 0, \qquad i = 1, 2, ..., 25 \qquad where$$

$$r_{14}(x) = L(x) - 305175780,$$

$$r_i(x) = R_i(x), \qquad i = 1, 13 \qquad \textit{and}$$

$$r_{i+1}(x) = R_i(x) \qquad i = 14, \ldots 24$$

Superlindo required 2438 iterations to solve the above LP problem with full accuracy. Because one cannot enter linear inequalities directly into Superlindo we entered the following problem: min L(x) subject to

$$r_i(x) \leq 0 \qquad 1 \leq i \leq 25.$$

Thus when phase I was finished the problem would terminate. This took 2236 iterations. Let

$$r(x) = Ax - b.$$

The components of A and b are listed at the end of this section.

A simple scheme of row scaling was used to convert to a better scaled problem. Let bm be the maximal component of the right hand side. bm=305175780. The scaled system is:

$$(bm/|b_i|)\, r_i(x) \leq 0 \qquad i = 1, 2, 3, \ldots, 14$$

$$bm\, r_i(x) \leq 0 \qquad i = 15, \ldots, 25$$

Thus 4.0 should be used. Some computations were made before 4.0 was coded using an ad hoc procedure of an overrelaxed gradient step to move off a degenerate point x. The step-length chosen was $100/\|H(x)\|$. This has worked for this problem in at most 3 steps for every example (over 100) tried. However the penalty function usually increases during these perturbations. Subsequently, the method of 4.0 was coded. The results were similar to those given below, with the moves in the null spaces (which we think of as gear shifts) replacing the gradient steps.

There follows 15 successive runs with random initial vectors with components lying between -1000 and 1000. The 12 numbers after x_0 are the components of the starting vector.

The last column contains a list of integers in order 1,2,3 interspersed with the symbol *. Each integer numbers a normal step of the algorithm while * indicates a gradient step. In what follows disregard this last column. We explain first the lines containing tt, info,

rmax, number, number, number. Here tt means the number of active residuals; its value is the first number. Info is a test for degeneracy; its value is the second number. If info = 0 the point is nondegenerate, info = 1 , degenerate. Rmax is the value of the maximum residual. It is the 3rd number.

Now, the line that contains step-length, gamma, number, number. Here step-length = $\|H^{-1}(x)\nabla f(x)\|$. Its value is the first number. The meaning of gamma is that of algorithm 3.0, the fraction of a Newton step. The value of gamma is the second number. At the bottom of the page we have x,r number,number. This the row-wise print out of the solution vector followed by the residuals at solution.

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APPENDIX THE COMPONENTS OF THE BLAIR MATRICES A & B

FIFTEEN RUNS WITH RANDOM STARTING POINTS

a(1,1)=-5.
a(2,1)=-1.
a(3,1)=0.
a(4,1)=0.
a(5,1)=0.
a(6,1)=0.
a(7,1)=0.
a(8,1)=0.
a(9,1)=0.
a(10,1)=0.
a(11,1)=0.0
a(12,1)=0.
a(13,1)=0.0
a(14,1)=5.
a(15,1) =-1.
a(16,1)=0.
a(17,1)=0.
a(18,1)=0.
a(19,1)=0.
a(20,1)=0.
a(21,1)=0.
a(22,1)=0.
a(23,1)=0.
a(24,1)=0.
a(25,1)=0.
a(1,2)=-10.
a(2,2)=-5.
a(3,2)=-1.
a(4,2)=0.
a(5,2)=0
a(6,2)=0.
a(7,2)=0.
a(8,2)=0.
a(9,2)=0.
a(10,2)=0.
a(11,2)=0.
a(12,2)=0.
a(13,2)=0.
a(14,2)=30.
a(15,2)=0.
a(16,2)=-1.
a(17,2)=0.
a(18,2)=0.
a(19,2)=0.
a(20,2)=0.
a(21,2)=0.
a(22,2)=0.
a(23,2)=0.
a(24,2)=0.
a(25,2)=0.
a(1,3)=-20.
a(2,3)=-10.
a(3,3)=-5.
a(4,3)=-1.
a(5,3)=0.
a(6,3)=0.
a(7,3)=0.
a(8,3)=0.
a(9,3)=0.
a(10,3)=0.
a(11,3)=0.
a(12,3)=0.
a(13,3)=0.
a(14,3)=155.
a(15,3)= 0.
a(16,3)=0.
a(17,3)=-1.
a(18,3)=0.
a(19,3)=0.
a(20,3)=0.
a(21,3)=0.
a(22,3)=0.
a(23,3)=0.
a(24,3)=0.
a(25,3)=0.
a(1,4)=-40.
a(2,4)=-20.
a(3,4)=-10.
a(4,4)=-5.
a(5,4)=-1.
a(6,4)=0.
a(7,4)=0.
a(8,4)=0.
a(9,4)=0.
a(10,4)=0.
a(11,4)=0.
a(12,4)=0.
a(13,4)=0.
a(14,4)=780.
a(15,4)=0.
a(16,4)=0.
a(17,4)=0.
a(18,4)=-1.
a(19,4)=0.
a(20,4)=0.
a(21,4)=0.
a(22,4)=0.
a(23,4)=0.
a(24,4)=0.
a(25,4)=0.
a(1,5)=-80.
a(2,5)=-40.
a(3,5)=-20.
a(4,5)=-10.
a(5,5)=-5.
a(6,5)=-1.
a(7,5)=0.
a(8,5)=0.
a(9,5)=0.
a(10,5)=0.
a(11,5)=0.
a(12,5)=0.
a(13,5)=0.
a(14,5)=3905.
a(15,5)=0.
a(16,5)=0.
a(17,5)=0.
a(18,5)=0.
a(19,5)=-1.
a(20,5)=0.
a(21,5)=0.
a(22,5)=0.
a(23,5)=0.
a(24,5)=0.
a(25,5)=0.
a(1,6)=-160.
a(2,6)=-80.
a(3,6)=-40.
a(4,6)=-20.
a(5,6)=-10.
a(6,6)=-5.
a(7,6)=-1.
a(8,6)= 0.
a(9,6)= 0.
a(10,6)=0.
a(11,6)=0.
a(12,6)=0.
a(13,6)=0.
a(14,6)=19530.
a(15,6)=0.
a(16,6)=0.
a(17,6)=0.
a(18,6)=0.
a(19,6)=0.
a(20,6)=-1.
a(21,6)=0.
a(22,6)=0.
a(23,6)=0.
a(24,6)=0.
a(25,6)=0.
a(1,7)=-320.
a(2,7)=-160.
a(3,7)=-80.
a(4,7)=-40.
a(5,7)=-20.
a(6,7)=-10.
a(7,7)=-5.
a(8,7)=-1.
a(9,7)= 0.
a(10,7)=0.
a(11,7)=0.
a(12,7)=0.
a(13,7)=0.
a(14,7)=97655.
a(15,7)=0.
a(16,7)=0.
a(17,7)=0.
a(18,7)=0.

a(19,7)=0.
a(20,7)=0.
a(21,7)=-1.
a(22,7)=0.
a(23,7)=0.
a(24,7)=0.
a(25,7)=0.
a(1,8)=-640.
a(2,8)=-320.
a(3,8)=-160.
a(4,8)=-80.
a(5,8)=-40.
a(6,8)=-20.
a(7,8)=-10.
a(8,8)=-5.
a(9,8)=-1.
a(10,8)=0.
a(11,8)=0.
a(12,8)=0.
a(13,8)=0.
a(14,8)=488280.
a(15,8)=0.
a(16,8)=0.
a(17,8)=0.
a(18,8)=0.
a(19,8)=0.
a(20,8)=0.
a(21,8)=0.
a(22,8)=-1.
a(23,8)=0.
a(24,8)=0.
a(25,8)=0.
a(1,9)=-1280.
a(2,9)=-640.
a(3,9)=-320.
a(4,9)=-160.
a(5,9)=-80.
a(6,9)=-40.
a(7,9)=-20.
a(8,9)=-10.
a(9,9)=-5
a(10,9)=-1.
a(11,9)=0.
a(12,9)=-1.
a(13,9)=0.
a(14,9)=2441405.
a(15,9)=0.
a(16,9)=0.
a(17,9)=0.
a(18,9)=0.
a(19,9)=0.
a(20,9)=0.
a(21,9)=0.
a(22,9)=0.
a(23,9)=-1.0
a(24,9)=0.
a(25,9)=0.
a(1,10)= -2560.
a(2,10)=-1280.
a(3,10)=-640.
a(4,10)=-320.
a(5,10)=-160.
a(6,10)=-80.
a(7,10)=-40.
a(8,10)=-20.
a(9,10)=-10.
a(10,10)=-5.
a(11,10)=-1.
a(12,10)=0.
a(13,10)=0.
a(14,10)=12207030.
a(15,10)=0.
a(16,10)=0.
a(17,10)=0.
a(18,10)=0.
a(19,10)=0.
a(20,10)=0.
a(21,10)=0.
a(22,10)=0.
a(23,10)=0.
a(24,10)=-1.
a(25,10)=0.
a(1,11)=-5120.
a(2,11)=-2560.
a(3,11)=-1280.
a(4,11)=-640.
a(5,11)=-320.
a(6,11)=-160.
a(7,11)=-80.
a(8,11)=-40.
a(9,11)=-20.
a(10,11)=-10.
a(11,11)=-5.
a(12,11)=-1.
a(13,11)=0.
a(14,11)=61035155.
a(15,11)=0.
a(16,11)=0.
a(17,11)=0.
a(18,11)=0.
a(19,11)=0.
a(20,11)=0.
a(21,11)=0.
a(22,11)=0.
a(23,11)=0.
a(24,11)=0.
a(25,11)=-1.
a(1,12)=-10240.
a(2,12)=-5120.
a(3,12)=-2560.
a(4,12)=-1280.
a(5,12)=-640.
a(6,12)=-320.
a(7,12)=-160.
a(8,12)=-80.
a(9,12)=-40.
a(10,12)=-20.
a(11,12)=-10.
a(12,12)=-5.
a(13,12)=-1.
a(14,12)=305175780.
a(15,12)=0.
a(16,12)=0.
a(17,12)=0.
a(18,12)=0.
a(19,12)=0.
a(20,12)=0.
a(21,12)=0.
a(22,12)=0.
a(23,12)=0.
a(24,12)=0.
a(25,12)=0.
b(1)=-4096.
b(2)=-2048.
b(3)=-1024.
b(4)=-512.
b(5)= -256.
b(6)=-128.
b(7)=-64.
b(8)=-32.
b(9)=-16.
b(10)=-8.
b(11)=-4.
b(12)=-2.
b(13)=-1.
b(14)=305175780.
b(15)=0.
b(16)=0.
b(17)=0.
b(18)=0.
b(19)=0.
b(20)=0.
b(21)=0.
b(22)=0.
b(23)=0.
b(24)=0.
b(25)=0.

```
RUN #1   11 Newton steps and   4   gradient steps.

x0    -157.239626709990         -743.322103338538         -971.979266891785
      507.770637071353          581.201493434374          -593.593439850401
     -677.746807236177         -361.133546071757           425.520516159178
      894.635221028124          563.412949712823           268.597848237653

tt,info,rmax,     7.00000000000000                    1      296624527029.611   *
tt,info,rmax,     14.0000000000000                    1      965325764497l.96   *

steplength,gamma     47765.5239230591       1.00000000000000          1
tt,info,rmax,     14.0000000000000                    0      2357700197.06739

steplength,gamma     19.2575641130305       1.00000000000000          2
tt,info,rmax,     12.0000000000000                    0      2428295204.71995

steplength,gamma     5454.13223736253     0.305175781250000D-00       3
tt,info,rmax,     14.0000000000000                    0      2428221099.03133

steplength,gamma     17.3705077615239       1.00000000000000          4
tt,info,rmax,     12.0000000000000                    0      1185139268.73383

steplength,gamma     5472.94705380250     0.762939453125000D-00       5
tt,info,rmax,     13.0000000000000                    0      1185130226.88878

tt,info,rmax,     1.00000000000000                    1      620675091.14548    *
tt,info,rmax,     19.0000000000000                    1      163062290989.794   *

steplength,gamma     198.732501363480       1.00000000000000          6
tt,info,rmax,     13.0000000000000                    0      153611593.072991

steplength,gamma     196.752514409908     0.152587890625000D-00       7
tt,info,rmax,     14.0000000000000                    0      153610594.293101

steplength,gamma     66.3907250047904     0.107421875000000D-001      8
tt,info,rmax,     13.0000000000000                    0      152948384.537403

steplength,gamma     83.3248672145210     0.131835937500000D-001      9
tt,info,rmax,     16.0000000000000                    0      152548504.421150

steplength,gamma   0.584973381718277        1.00000000000000          10
tt,info,rmax,     12.0000000000000                    0      133329412.581193

steplength,gamma   0.504481166507457        1.00000000000000          11
tt,info,rmax,     12.0000000000000                    0    0.250339508056641D-005

steplength,gamma   0.936594908882387D-014   1.00000000000000
tt,info,rmax,     12.0000000000000                    0    0.000000000000000

 x . r      0.165629975217302D-031    0.608593862426366D-031
   0.175644810928116D-030    0.412919380076623D-030    0.850490663441403D-030
   0.173795918181504D-029    0.345126646034193D-029    0.690253292068385D-029
   0.136078506150625D-028    0.268212707775144D-028    0.544314024602498D-028
    1.00000000000000         -457763664.000000         -457763664.000000
   -457763664.000000         -457763664.000000         -457763664.000000
   -457763664.000000         -457763664.000000         -457763664.000000
   -457763664.000000         -457763664.000000         -457763664.000000
   -457763664.000000         0.000000000000000         0.000000000000000
  -0.505462562158010D-023   -0.185728104234804D-022   -0.536025414753610D-022
  -0.126012992240322D-021   -0.259549148196485D-021   -0.530383041966730D-021
  -0.105324292021762D-020   -0.210648584043524D-020   -0.415278637114376D-020
  -0.818520212283408D-020   -0.166111454845750D-019
```

```
RUN #2              7 Newton steps and  3  gradient steps

x0    -930.292890085242        -863.844227490599          -322.317913607396
       244.628299272343         719.519989562319           543.254998043481
      -508.677056980336        -162.246891100147           618.731966442102
       416.323740083475        -708.495647012945           253.088220718418
tt,info,rmax,     8.00000000000000                  1      283902854639.046   *
tt,info,rmax,     1.00000000000000                  1      137111628067.87    *
tt,info,rmax,     16.0000000000000                  1      35977493981424.    *

steplength,gamma     435728.172539526        1.00000000000000              1
tt,info,rmax,     12.0000000000000                  0      161170658.157925

steplength,gamma     10514.7126566587        0.274658203125000D-003        2
tt,info,rmax,     14.0000000000000                  0      161126391.314559

steplength,gamma     60.4510944982947        0.312500000000000D-001        3
tt,info,rmax,     14.0000000000000                  0      156091441.193965

steplength,gamma     52.5504577006773        0.125000000000000             4
tt,info,rmax,     15.0000000000000                  0      148358483.404517

steplength,gamma     126.089758342685        0.644531250000000D-001        5
tt,info,rmax,     14.0000000000000                  0      147307230.642514

steplength,gamma    0.649075995843773        1.00000000000000              6
tt,info,rmax,     12.0000000000000                  0      129555772.393379

steplength,gamma    0.490203091522860        1.00000000000000              7
tt,info,rmax,     12.0000000000000                  0    0.244379043579102D-005

steplength,gamma    0.913113886853910D-014   1.00000000000000
tt,info,rmax,     12.0000000000000                  0    0.00000000000000

 x , r      0.331259950434605D-031    0.100148357108136D-030
    0.200296714216273D-030   0.409837892165604D-030   0.819675784331207D-030
    0.167632942359465D-029   0.337731075047746D-029   0.680392530753123D-029
    0.134106353887572D-028   0.264268403249039D-028   0.528536806498078D-028
     1.00000000000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        0.00000000000000          0.00000000000000
   -0.101092512431602D-022  -0.305628525956006D-022  -0.611257051912012D-022
   -0.125072596775842D-021  -0.250145193551685D-021  -0.511575132677130D-021
   -0.103067342307010D-020  -0.207639318557168D-020  -0.409260106141704D-020
   -0.306483150338064D-020  -0.161290620067613D-019
```

```
RUN   #3          7 Newton steps and 1  gradient step


x0     384.636186040327          521.532891478957          -700.058083640171
      890.556864074000          22.8991136970037          -639.569422439330
      823.899904502448          823.568592993691           996.340118297574
     -94.6125941635709         -466.486448786907          -450.878854236014

tt,info,rmax,      17.0000000000000                    0      52946725930l.064   *


steplength,gamma      2212.27366081213          1.00000000000000               1
tt,info,rmax,      14.0000000000000                    0      194786857.815711

steplength,gamma      11.5597435264857         0.937500000000000D-001          2
tt,info,rmax,      15.0000000000000                    0      179212381.151034

steplength,gamma      17.4100728479677         0.312500000000000D-001          3
tt,info,rmax,      15.0000000000000                    0      173957718.602083

steplength,gamma      13.1910259009825         0.109375000000000               4
tt,info,rmax,      16.0000000000000                    0      156186399.016667

steplength,gamma      4.83788792413538         0.375000000000000               5
tt,info,rmax,      17.0000000000000                    0      126699425.005225

steplength,gamma     0.654804890885739          1.00000000000000               6
tt,info,rmax,      12.0000000000000                    0      134111089.546827

steplength,gamma     0.507437943699474          1.00000000000000               7
tt,info,rmax,      12.0000000000000                    0     0.250339508056641D-005

steplength,gamma     0.936796982418018D-014     1.00000000000000
tt,info,rmax,      12.0000000000000                    0     0.000000000000000

 x , r     0.130963236218332D-031    0.485334345985583D-031
    0.140207699951391D-030   0.372860037233369D-030   0.850490663441403D-030
    0.173795918181504D-029   0.345126646034193D-029   0.690253292068385D-029
    0.138050658413677D-028   0.268212707775144D-028   0.544314024602493D-028
     1.00000000000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        0.000000000000000         0.000000000000000
  -0.399668072404008D-023  -0.148112285655603D-022  -0.427879936338408D-022
  -0.113787851202082D-021  -0.259549148196485D-021  -0.530383041966730D-021
  -0.105324292021762D-020  -0.210648584043524D-020  -0.421297168087048D-020
  -0.818520212283408D-020  -0.166111454845750D-019
```

```
RUN #4            7 Newton steps


x0    -732.342233747725           741.538604529571           244.383834286300
     341.271154817054          324.279857033038          112.113379925855
     262.873709250308          604.307441254551         -321.373886549000
    -966.821403447874         -693.073136470057         -623.141231756370

tt,info,rmax,     17.0000000000000                    0      109622739862 3.85    1

steplength,gamma      2002.80968276938         1.00000000000000                  2
tt,info,rmax,     13.0000000000000                    0      153540383.712848

steplength,gamma      130.062162441505        0.634765625000000D-002            3
tt,info,rmax,     14.0000000000000                    0      153117350.272783

steplength,gamma      29.8103139169651        0.253906250000000D-001            4
tt,info,rmax,     15.0000000000000                    0      151769052.907618

steplength,gamma      7.61286793105905        0.937500000000000D-001            5
tt,info,rmax,     16.0000000000000                    0      149354307.684918

steplength,gamma    0.480797575295140          1.00000000000000                  6
tt,info,rmax,     12.0000000000000                    0      133329412.581193

steplength,gamma    0.504481166507457          1.00000000000000                  7
tt,info,rmax,     12.0000000000000                    0    0.244379043579102D-005

steplength,gamma    0.918104743739594D-014     1.00000000000000
tt,info,rmax,      1.00000000000000                   1    0.596046447753906D-007

 x  .  r     0.770371977754894D-033    0.433334237487128D-032
    0.208000433993821D-031   0.962964972193618D-031   0.440652771275600D-030
    0.202145606962884D-029   0.917050802319426D-029   0.410207670714926D-028
    0.183015730011275D-027   0.795171792462780D-027   0.338263556158770D-026
     1.00000000000002        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        0.596046447753906D-007  -0.596046447753906D-007
   -0.235098866120005D-024  -0.132243112192503D-023  -0.634766938524012D-023
   -0.293873582650006D-022  -0.134476551420643D-021  -0.616899424698892D-021
   -0.279861690229253D-020  -0.125185444231580D-019  -0.558519674263973D-019
   -0.242667168818140D-018  -0.103229843243272D-017
```

RUN #5   7 Newton steps

```
x0   -114.147717658216         558.470022662264         -490.039243319173
      624.461662419657        -654.960097720481         -156.588812028271
       98.7877842091593        675.860880011940          297.942678453505
     -427.420645454674        -607.602961981098         -603.144542822476

tt,info,rmax,     19.0000000000000                  0     762408473253.473

steplength,gamma     1696.80530559346         1.00000000000000          1
tt,info,rmax,     13.0000000000000                  0     194664482.932851

steplength,gamma     78.7277480487342         0.122070312500000D-003    2
tt,info,rmax,     15.0000000000000                  0     194642039.668371

steplength,gamma     10.5301245257422         0.125000000000000         3
tt,info,rmax,     16.0000000000000                  0     171746426.699385

steplength,gamma     8.22887713006761         0.156250000000000D-001    4
tt,info,rmax,     16.0000000000000                  0     169258335.804595

steplength,gamma     4.40534461134211         0.312500000000000         5
tt,info,rmax,     18.0000000000000                  0     125867764.378646

steplength,gamma   0.637037280856529          1.00000000000000          6
tt,info,rmax,     12.0000000000000                  0     134638584.393735

steplength,gamma   0.509430809427019          1.00000000000000          7
tt,info,rmax,     12.0000000000000                  0   0.250339508056641D-005

steplength,gamma   0.936745800551893D-014     1.00000000000000
tt,info,rmax,     12.0000000000000                  0   0.000000000000000

 x . r     0.108815041857879D-031    0.396741568543771D-031
   0.117866912596499D-030   0.311230279012977D-030   0.764209001932655D-030
   0.173795918181504D-029   0.350057026691824D-029   0.700114053383648D-029
   0.138050658413677D-028   0.268212707775144D-028   0.544314024602498D-028
    1.00000000000000        -457763664.000000         -457763664.000000
   -457763664.000000        -457763664.000000         -457763664.000000
   -457763664.000000        -457763664.000000         -457763664.000000
   -457763664.000000        -457763664.000000         -457763664.000000
   -457763664.000000         0.000000000000000         0.000000000000000
  -0.332077148394506D-023  -0.121075916051802D-022  -0.359701265163607D-022
  -0.949799419124819D-022  -0.233218075191045D-021  -0.530383041966730D-021
  -0.106828924764930D-020  -0.213657849529860D-020  -0.421297166067048D-020
  -0.818520212283408D-020  -0.166111454845750D-019
```

RUN #6   7 Newton steps and 2 gradient steps

```
x0    472.361079681048        -831.003678005437       -740.935969423671
     899.660706261435         814.735410238548        -652.488692051425
     64.0326036041202        -59.8063268680018         72.9986403563025
     955.477280014122         640.867128180124         712.585229961250

tt,info,rmax,     5.00000000000000                1     268083811943.665   *
tt,info,rmax,     16.0000000000000                1     23715049130988.6   *

steplength,gamma     50637.9972940535       1.00000000000000          1
tt,info,rmax,     12.0000000000000                0     161169141.153651

steplength,gamma     10514.7126636119     0.274658203125000D-003      2
tt,info,rmax,     14.0000000000000                0     161124874.726943

steplength,gamma     60.4510874755542     0.312500000000000D-001      3
tt,info,rmax,     14.0000000000000                0     156089971.999712

steplength,gamma     52.5504645937000     0.125000000000000           4
tt,info,rmax,     15.0000000000000                0     148358479.968610

steplength,gamma     126.089752134237     0.644531250000000D-001      5
tt,info,rmax,     14.0000000000000                0     147307227.428061

steplength,gamma   0.649075033909147        1.00000000000000          6
tt,info,rmax,     12.0000000000000                0     129555772.393379

steplength,gamma   0.490203091522860        1.00000000000000          7
tt,info,rmax,     12.0000000000000                0   0.244379043579102D-005

steplength,gamma   0.911924546507179D-014   1.00000000000000
tt,info,rmax,     12.0000000000000                0   0.00000000000000

 x , r     0.331259950434605D-031   0.986076131526265D-031
   0.200296714216273D-030   0.406756404254584D-030   0.819675784331207D-030
   0.167632942359465D-029   0.332800694390114D-029   0.670531769437860D-029
   0.134106353887572D-028   0.264268403249039D-028   0.520648197445868D-028
    1.00000000000000        -457763664.000000        -457763664.000000
   -457763664.000000        -457763664.000000        -457763664.000000
   -457763664.000000        -457763664.000000        -457763664.000000
   -457763664.000000        -457763664.000000        -457763664.000000
   -457763664.000000         0.00000000000000         0.00000000000000
  -0.101092512431602D-022  -0.300926548633606D-022  -0.611257051912012D-022
  -0.124132201311362D-021  -0.250145193551685D-021  -0.511575132677130D-021
  -0.101562710163842D-020  -0.204630053070852D-020  -0.409260106141704D-020
  -0.806463150338064D-020  -0.158889217678544D-019
```

RUN #7   8 Newton steps

```
x0   -980.614385562064        754.663860379469        -849.966118423708
      872.092335185815        903.372305432807         903.822716644623
     -787.423383854608       -889.882406466205        -656.189370753424
     -801.412660064975       -257.581608353113        -733.888854017488

tt,info,rmax,     20.0000000000000                   0     930810709180.125

steplength,gamma     2781.62252337554        1.00000000000000             1
tt,info,rmax,     12.0000000000000                   0     153611829.496706

steplength,gamma     3648.13836926195      0.391006469726563D-004         2
tt,info,rmax,     15.0000000000000                   0     153605823.174790

steplength,gamma     64.0782140937533      0.830078125000000D-002         3
tt,info,rmax,     16.0000000000000                   0     153096180.228943

steplength,gamma     27.6115425905722      0.195312500000000D-001         4
tt,info,rmax,     15.0000000000000                   0     152568067.708317

steplength,gamma     73.2258353230729      0.305175781250000D-004         5
tt,info,rmax,     15.0000000000000                   0     152567395.728954

steplength,gamma   0.340599172069112        1.00000000000000              6
tt,info,rmax,     12.0000000000000                   0     132047268.790020

steplength,gamma   0.499630146021219        1.00000000000000              7
tt,info,rmax,     12.0000000000000                   0   0.250339508056641D-005

steplength,gamma   0.934835255493608D-014   1.00000000000000              8
tt,info,rmax,     12.0000000000000                   0   0.000000000000000

 x , r     0.215704153771370D-031    0.801186856865090D-031
     0.206459690038312D-030    0.412919380076623D-030    0.844327687619364D-030
     0.168865537523873D-029    0.342661455705377D-029    0.680392530753123D-029
     0.136078506150625D-028    0.268212707775144D-028    0.536425415550288D-028
      1.00000000000000         -457763664.000000         -457763664.000000
     -457763664.000000         -457763664.000000         -457763664.000000
     -457763664.000000         -457763664.000000         -457763664.000000
     -457763664.000000         -457763664.000000         -457763664.000000
     -457763664.000000         0.000000000000000         0.000000000000000
    -0.658276825136013D-023   -0.244502820764805D-022   -0.630064961201612D-022
    -0.126012992240322D-021   -0.257668357267525D-021   -0.515336714535050D-021
    -0.104571975650178D-020   -0.207639318557188D-020   -0.415278637114376D-020
    -0.818520212283408D-020   -0.163704042456682D-019
```

RUN #8 9 Newton steps

```
x0     489.859138456204        20.9052331088909        -633.600054840564
      174.179628824158       -653.211700218227         675.884989041090
      425.973625697280       -171.141846780673        -754.508656207635
      955.040074874622       -896.536088135900         206.621797933154

tt,info,rmax,      18.0000000000000                   0      273601096408.878

steplength,gamma      2023.42010222668        1.00000000000000             1
tt,info,rmax,      12.0000000000000                   0      159079979.545477

steplength,gamma      15612.1105994219       0.190734863281250D-005        2
tt,info,rmax,      14.0000000000000                   0      159079676.124495

steplength,gamma      23.5758796432499       0.781250000000000D-002        3
tt,info,rmax,      16.0000000000000                   0      158553666.463820

steplength,gamma      18.9507539629753       0.125000000000000             4
tt,info,rmax,      15.0000000000000                   0      154534990.248806

steplength,gamma      34.1524046774300       0.244140625000000D-003        5
tt,info,rmax,      16.0000000000000                   0      154529135.198641

steplength,gamma      92.0011728271753       0.280761718750000D-001        6
tt,info,rmax,      14.0000000000000                   0      153934008.178371

steplength,gamma     0.213780894549538        1.00000000000000             7
tt,info,rmax,      12.0000000000000                   0      129555772.393379

steplength,gamma     0.490203091522860        1.00000000000000             8
tt,info,rmax,      12.0000000000000                   0    0.238418579101563D-005

steplength,gamma     0.895106789666395D-014   1.00000000000000             9
tt,info,rmax,      1.00000000000000                   1    0.596046447753906D-007

 x , r     0.776390508831104D-033    0.433334237487128D-032
   0.206074504049434D-031    0.962964972193618D-031    0.446815747097839D-030
   0.204610797291700D-029    0.917050802319426D-029    0.410207670714926D-028
   0.183015730011275D-027    0.795171792462780D-027    0.343312265952184D-026
    1.00000000000002         -457763664.000000         -457763664.000000
   -457763664.000000         -457763664.000000         -457763664.000000
   -457763664.000000         -457763664.000000         -457763664.000000
   -457763664.000000         -457763664.000000         -457763664.000000
   -457763664.000000         0.596046447753906D-007   -0.596046447753906D-007
  -0.236935576011567D-024   -0.132243112192503D-023   -0.628889466871012D-023
  -0.293873582650006D-022   -0.136357342349603D-021   -0.624422588414732D-021
  -0.279861690229253D-020   -0.125185444231580D-019   -0.558519674263973D-019
  -0.242667168818140D-018   -0.104770587172276D-017
```

```
RUN  #9        10 Newton steps and 2 gradient step

x0   -834.972506153810        -983.769175140755         447.075901089682
    -875.078848947569         924.303418741492         38.9442774086471
    -795.176371801571        -0.500703910808475        487.150834676203
    -346.782467450234         322.616508754233         721.511346799489
tt,info,rmax,    7.00000000000000                 1     30022521428.460   *
tt,info,rmax,    15.0000000000000                 1     1735910335087O.1   *

steplength,gamma     54027.8046190082      1.00000000000000              1
tt,info,rmax,    12.0000000000000                 0     1786211071.36421

steplength,gamma     15776.5628664350     0.316619873046875D-003         2
tt,info,rmax,    14.0000000000000                 0     1785645521.44156

steplength,gamma     27.7719244099574     0.250000000000000              3
tt,info,rmax,    14.0000000000000                 0     1339238345.83148

steplength,gamma     8.49827918067834      1.00000000000000              4
tt,info,rmax,    12.0000000000000                 0     590430919.804070

steplength,gamma     2736.52420367105     0.762939453125000D-005         5
tt,info,rmax,    13.0000000000000                 0     590426415.173640

steplength,gamma     47.6924304507833     0.781250000000000D-002         6
tt,info,rmax,    14.0000000000000                 0     585835987.155582

steplength,gamma     8.13685607249363     0.625000000000000D-001         7
tt,info,rmax,    15.0000000000000                 0     549466999.076904

steplength,gamma     42.5352574148447     0.328125000000000              8
tt,info,rmax,    16.0000000000000                 0     370527291.858912

steplength,gamma     1.55797420666023      1.00000000000000              9
tt,info,rmax,    12.0000000000000                 0     133329412.581193

steplength,gamma    0.504481166507457      1.00000000000000              10
tt,info,rmax,    12.0000000000000                 0   0.250339508056641D-005

steplength,gamma    0.936377296560471D-014     1.00000000000000
tt,info,rmax,    12.0000000000000                 0   0.000000000000000
 x , r     0.165629975217302D-031   0.597038282760043D-031
    0.175644810928116D-030   0.422163843809682D-030   0.844327687619364D-030
    0.173795918181504D-029   0.342661455705377D-029   0.690253292068385D-029
    0.138050658413677D-028   0.268212707775144D-028   0.544314024602498D-028
     1.00000000000000        -457763664.000000        -457763664.000000
    -457763664.000000        -457763664.000000        -457763664.000000
    -457763664.000000        -457763664.000000        -457763664.000000
    -457763664.000000        -457763664.000000        -457763664.000000
    -457763664.000000        0.000000000000000        0.000000000000000
   -0.505462562158010D-023  -0.182201621243004D-022  -0.536025414753610D-022
   -0.128834178633763D-021  -0.257668357267525D-021  -0.530383041966730D-021
   -0.104571975650178D-020  -0.210648584043524D-020  -0.421297168087048D-020
   -0.818520212283408D-020  -0.166111454845750D-019
```

RUN #10 7 Newton steps

```
xO     168.562088607904       879.936512051979       282.756764008947
     -215.631795284871       386.541205751119      -272.694289835621
     -312.386754660958      -847.804830474609      -918.030969523965
     -783.756813578536      -462.177283109723       318.376253985250

tt,info,rmax,       18.0000000000000                 0     280160813516.508

steplength,gamma     1935.29079830120       1.00000000000000          1
tt,info,rmax,       12.0000000000000                 0     851810879.278383

steplength,gamma     7019.68685306682       0.122070312500000D-003     2
tt,info,rmax,       14.0000000000000                 0     851706898.458158

steplength,gamma     78.7484599706272       0.781250000000000D-001     3
tt,info,rmax,       13.0000000000000                 0     785166074.844281

steplength,gamma     335.697096172363       0.209960937500000D-001     4
tt,info,rmax,       14.0000000000000                 0     768681659.969552

steplength,gamma     3.57321680138793       1.00000000000000          5
tt,info,rmax,       12.0000000000000                 0     129555772.393379

steplength,gamma     0.490203091522860      1.00000000000000          6
tt,info,rmax,       12.0000000000000                 0   0.244379043579102D-005

steplength,gamma     0.913177239355039D-014  1.00000000000000         7
tt,info,rmax,       12.0000000000000                 0   0.000000000000000

 x , r    0.325482160601443D-031   0.986076131526265D-031
     0.200296714216273D-030   0.409837892165604D-030   0.819675784331207D-030
     0.167632942359465D-029   0.332800694390114D-029   0.680392530753123D-029
     0.134106353887572D-028   0.264268403249039D-028   0.528536806498078D-028
      1.00000000000000        -457763664.000000        -457763664.000000
     -457763664.000000        -457763664.000000        -457763664.000000
     -457763664.000000        -457763664.000000        -457763664.000000
     -457763664.000000        -457763664.000000        -457763664.000000
     -457763664.000000        0.000000000000000        0.000000000000000
    -0.993292709357019D-023  -0.300926548633606D-022  -0.611257051912012D-022
    -0.125072596775842D-021  -0.250145193551685D-021  -0.511575132677130D-021
    -0.101562710163842D-020  -0.207639318557188D-020  -0.409260106141704D-020
    -0.806483150338064D-020  -0.161296630067613D-019
```

```
RUN  #11          10 Newton steps and 2 gradient step

x0    -507.462338260756         -745.331526294334           822.626989288406
     460.053875018011         -77.6430245183148          -406.981993993563
    -554.345688116258        -506.427349605299           53.2199324551891
    -905.916211360387        -924.250809940748           901.741966752070
tt,info,rmax,         9.00000000000000                    1      282058958142.296   *
tt,info,rmax,         14.0000000000000                    1      9678403149694.32   *

steplength,gamma       45364.5042318625           1.00000000000000                 1
tt,info,rmax,         12.0000000000000                    0      429107069.641468

steplength,gamma       15773.5499860498         0.748634338378906D-004             2
tt,info,rmax,         14.0000000000000                    0      429074945.212750

steplength,gamma       29.2671518584511         0.625000000000000D-001             3
tt,info,rmax,         14.0000000000000                    0      402258848.273416

steplength,gamma       10.3875006835552         0.375000000000000                  4
tt,info,rmax,         18.0000000000000                    0      295236784.148774

steplength,gamma       2.41364909828324           1.00000000000000                 5
tt,info,rmax,         13.0000000000000                    0      287724616.031283

steplength,gamma       106.889816604460         0.244140625000000D-003             6
tt,info,rmax,         16.0000000000000                    0      287655027.217895

steplength,gamma       9.48881403921367         0.156250000000000D-001             7
tt,info,rmax,         18.0000000000000                    0      283221899.663543

steplength,gamma       11.5704315204349         0.156250000000000D-001             8
tt,info,rmax,         19.0000000000000                    0      278861295.027645

steplength,gamma     0.911987664675868            1.00000000000000                 9
tt,info,rmax,         12.0000000000000                    0      135022355.068402

steplength,gamma     0.510872132671297            1.00000000000000                 10
tt,info,rmax,         12.0000000000000                    0    0.250339508056641D-005

steplength,gamma     0.935628649287364D-014      1.00000000000000
tt,info,rmax,         12.0000000000000                    0    0.000000000000000
 x , r      0.938890847888777D-032   0.337037740267766D-031
   0.100148357108136D-030   0.263467216392174D-030   0.659438412958189D-030
   0.156539585879795D-029   0.350057026691824D-029   0.700114053383648D-029
   0.138050658413677D-028   0.268212707775144D-028   0.536425415550288D-028
    1.00000000000000         -457763664.000000          -457763664.000000
   -457763664.000000         -457763664.000000          -457763664.000000
   -457763664.000000         -457763664.000000          -457763664.000000
   -457763664.000000         -457763664.000000          -457763664.000000
   -457763664.000000         0.000000000000000          0.000000000000000
  -0.286526743083756D-023  -0.102855753927502D-022  -0.305628525956006D-022
  -0.804038122130416D-022  -0.201244629398724D-021  -0.477720895955849D-021
  -0.106828924764930D-020  -0.213657849529860D-020  -0.421297168087048D-020
  -0.818520212283408D-020  -0.163704042456682D-019
```

```
RUN   #12          10 Newton steps


x0    -415.465089220958          93.3967863568357          -655.692275827660
   -777.129532256195         503.520149216688         -632.224189726941
   -976.113784378833         686.576335880251          263.850657800655
   -78.1654829784306         169.694047289757         -953.766508610371

tt,info,rmax,      19.0000000000000                        0     669201674021.751

steplength,gamma     2075.58548391905          1.00000000000000             1
tt,info,rmax,      12.0000000000000                        0     495718564.127140

steplength,gamma     4.19354626531279          0.500000000000000            2
tt,info,rmax,      13.0000000000000                        0     356977880.225168

steplength,gamma     581.479889666029          0.610351562500000D-004       3
tt,info,rmax,      14.0000000000000                        0     356957986.239844

steplength,gamma     17.0801205817469          0.781250000000000D-001       4
tt,info,rmax,      17.0000000000000                        0     331543988.701578

steplength,gamma     6.52560684331984          0.500000000000000            5
tt,info,rmax,      15.0000000000000                        0     181610652.924412

steplength,gamma     2.39912596957116          0.125000000000000            6
tt,info,rmax,      15.0000000000000                        0     162997667.387076

steplength,gamma     18.3964281018146          0.218750000000000            7
tt,info,rmax,      19.0000000000000                        0     134566511.091070

steplength,gamma   0.662869275097632           1.00000000000000             8
tt,info,rmax,      12.0000000000000                        0     135022355.068401

steplength,gamma   0.510872132671297           1.00000000000000             9
tt,info,rmax,      12.0000000000000                        0   0.250339508056641D-005

steplength,gamma   0.938228545189278D-014      1.00000000000000             10
tt,info,rmax,      12.0000000000000                        0   0.000000000000000
  x , r      0.938890847888777D-032   0.342815530100928D-031
   0.100918729085891D-030   0.268089448258703D-030   0.659438412958189D-030
   0.156539585879795D-029   0.350057026691824D-029   0.700114053383648D-029
   0.138050658413677D-028   0.272157012301249D-028   0.544314024602498D-028
    1.00000000000000        -457763664.000000        -457763664.000000
   -457763664.000000        -457763664.000000        -457763664.000000
   -457763664.000000        -457763664.000000        -457763664.000000
   -457763664.000000        -457763664.000000        -457763664.000000
   -457763664.000000        0.000000000000000        0.000000000000000
  -0.286526743083756D-023  -0.104618995423402D-022  -0.307979514617206D-022
  -0.818144054097616D-022  -0.201244629398724D-021  -0.477720895955849D-021
  -0.106828924764930D-020  -0.213657849529860D-020  -0.421297168087048D-020
  -0.830557274228752D-020  -0.166111454845750D-019
  PAUSE?
```

RUN #13   6 Newton steps 2 gradient steps

```
x0     536.477440189505          961.442495391034          449.092889867492
    -565.222855711304         180.182538508556          768.913749734951
    -80.3906713861346        -628.929442909511          273.249274237714
     157.383786486445          42.1332745428034         528.871667806834

tt,info,rmax,     4.00000000000000                  1      191934030789.158  *
tt,info,rmax,     15.0000000000000                  1      16845691099344.1  *

steplength,gamma     37117.6105912337        1.00000000000000           1
tt,info,rmax,     14.0000000000000                  0      939465467.781931

steplength,gamma     11.6912012042131       0.500000000000000           2
tt,info,rmax,     14.0000000000000                  0      532153589.895607

steplength,gamma     21.0876406660046       0.500000000000000           3
tt,info,rmax,     15.0000000000000                  0      283305905.447369

steplength,gamma     75.6961499181736       0.226562500000000           4
tt,info,rmax,     16.0000000000000                  0      219586876.653159

steplength,gamma     1.44623443931224        1.00000000000000           5
tt,info,rmax,     12.0000000000000                  0      133329412.581193

steplength,gamma   0.504481166507457         1.00000000000000           6
tt,info,rmax,     12.0000000000000                  0   0.250339508056641D-005

steplength,gamma   0.934252011646448D-014    1.00000000000000
tt,info,rmax,     12.0000000000000                  0   0.000000000000000

 x , r     0.161778115328528D-031   0.597038282760043D-031
    0.174104066972606D-030   0.422163843809682D-030   0.844327687619364D-030
    0.173795918181504D-029   0.342661455705377D-029   0.690253292068385D-029
    0.136078506150625D-028   0.268212707775144D-028   0.528536806498073D-028
     1.00000000000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000        -457763664.000000         -457763664.000000
    -457763664.000000         0.000000000000000         0.000000000000000
   -0.493707618852010D-023  -0.182201621243004D-022  -0.531323437431210D-022
   -0.128834178633763D-021  -0.257668357267525D-021  -0.530383041966730D-021
   -0.104571975650178D-020  -0.210648584043524D-020  -0.415278637114376D-020
   -0.818520212283408D-020  -0.161296630067613D-019
```

RUN #14 6 Newton steps 3 gradient steps

```
x0     -313.188277293696        -74.7127697490351         363.859180360679
        225.308342001151       -449.303067641651        -722.721760127721
        749.187281293977        334.306731674827         566.136306586745
        21.6374771325915        834.136420883962        -471.498311508622

tt,info,rmax,      8.00000000000000                1      220557173979.063   *
tt,info,rmax,      1.00000000000000                1      2819394490440.15   *
tt,info,rmax,      17.0000000000000                1      743317395805483.   *

steplength,gamma     895425.145414435        1.00000000000000             1
tt,info,rmax,      13.0000000000000                0      153587264.635352

steplength,gamma     98.7784435622585      0.732421875000000D-002         2
tt,info,rmax,      15.0000000000000                0      153108097.868560

steplength,gamma     13.9259737920949      0.117187500000000D-001         3
tt,info,rmax,      14.0000000000000                0      152485904.282359

steplength,gamma     16.3951704458082      0.625000000000000D-001         4
tt,info,rmax,      15.0000000000000                0      150818862.186507

steplength,gamma   0.717112686602673         1.00000000000000             5
tt,info,rmax,      12.0000000000000                0      132047268.790020

steplength,gamma   0.499630146021219         1.00000000000000             6
tt,info,rmax,      12.0000000000000                0   0.244379043579102D-005

steplength,gamma   0.915426311736677D-014    1.00000000000000
tt,info,rmax,      1.00000000000000                1   0.596046447753906D-007

  x , r     0.770371977754894D-033    0.430926825056644D-032
     0.208000433993821D-031    0.962964972193618D-031    0.440652771275800D-030
     0.202145606962884D-029    0.917050802319426D-029    0.410207670714926D-028
     0.183015730011275D-027    0.795171792462780D-027    0.338263556158770D-026
      1.00000000000002         -457763664.000000         -457763664.000000
     -457763664.000000         -457763664.000000         -457763664.000000
     -457763664.000000         -457763664.000000         -457763664.000000
     -457763664.000000         -457763664.000000         -457763664.000000
     -457763664.000000         0.596046447753906D-007   -0.596046447753906D-007
   -0.235098866120005D-024   -0.131508428235878D-023   -0.634766938524012D-023
   -0.293873582650006D-022   -0.134476551420643D-021   -0.616899424698892D-021
   -0.279861690229253D-020   -0.125185444231580D-019   -0.558519674263973D-019
   -0.242667168818140D-018   -0.103229843243272D-017
```

RUN #15 8 Newton steps

```
x0    296.981125772494       805.619158814028        51.6047080018680
     -40.5170014433711      -670.077290135360      -975.512313377759
      78.1184786292034      -737.687608386972        -6.90543145820155
      95.5765248137808      -613.386207147848      -628.146325780695

tt,info,rmax,       19.0000000000000                  0     744985996579.413

steplength,gamma      1863.45202256309          1.00000000000000             1
tt,info,rmax,       14.0000000000000                  0     151289322.544588

steplength,gamma      20.6857895261084        0.253906250000000D-001         2
tt,info,rmax,       15.0000000000000                  0     150658392.083229

steplength,gamma      2.08038227471001        0.250000000000000              3
tt,info,rmax,       16.0000000000000                  0     144578667.929296

steplength,gamma      35.9811215440998        0.488281250000000D-003         4
tt,info,rmax,       18.0000000000000                  0     144551342.269987

steplength,gamma      6.59190252887505        0.781250000000000D-002         5
tt,info,rmax,       19.0000000000000                  0     144407436.410621

steplength,gamma     0.455879229119055          1.00000000000000             6
tt,info,rmax,       12.0000000000000                  0     135022355.068402

steplength,gamma     0.510872132671297          1.00000000000000             7
tt,info,rmax,       12.0000000000000                  0   0.250339508056641D-00

steplength,gamma     0.938331531860259D-014     1.00000000000000             8
tt,info,rmax,       12.0000000000000                  0   0.000000000000000

  x , r     0.938890847888777D-032   0.342815530100928D-031
    0.100918729085891D-030   0.268089448258703D-030   0.671764364602268D-03
    0.156539585879795D-029   0.350057026691824D-029   0.700114053383648D-02
    0.140022810676730D-028   0.272157012301249D-028   0.544314024602498D-02
     1.00000000000000        -457763664.000000        -457763664.000000
    -457763664.000000        -457763664.000000        -457763664.000000
    -457763664.000000        -457763664.000000        -457763664.000000
    -457763664.000000        -457763664.000000        -457763664.000000
    -457763664.000000        0.000000000000000        0.000000000000000
   -0.286526743083756D-023  -0.104618995423402D-022  -0.307979514617206D-02
   -0.818144054097616D-022  -0.205006211256644D-021  -0.477720895955849D-02
   -0.106828924764930D-020  -0.213657849529860D-020  -0.427315699059720D-02
   -0.830557274228752D-020  -0.166111454845750D-019
```

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